

# Differential relations for almost Belyi maps

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## Abstract

Several kinds of differential relations for polynomial components of almost Belyi maps are presented. Saito's theory of free divisors give particularly interesting (yet conjectural) logarithmic action of vector fields. The differential relations implied by Kitaev's construction of algebraic Painlevé VI solutions through pull-back transformations are used to compute almost Belyi maps for the pull-backs giving all genus 0 and 1 Painlevé VI solutions in the Lisovyy-Tykhyy classification.

## 1 Introduction

Importance of Belyi maps was highlighted in the *l'Esquisse d'une programme* by Grothendieck [9]. Since then, Belyi maps attract increasing attention in algebraic geometry, number theory, mathematical physics. One elementary application of Belyi maps is pull-back transformations of hypergeometric differential equations to Fuchsian equations with a small number of singularities, and corresponding transformations of special functions [12], [28], [29].

Recall that a *Belyi map* is an algebraic covering  $\varphi : C \rightarrow \mathbb{P}^1$  that branches only above  $\{0, 1, \infty\} \subset \mathbb{P}^1$ . In particular, a genus 0 Belyi map (with  $C \cong \mathbb{P}^1$ ) is defined by a rational function  $\varphi(x) \in \mathbb{C}(x)$  such that all branching points  $\{x : \varphi'(x) = 0\}$  lie in the fibers  $\varphi(x) \in \{0, 1, \infty\}$ .

Almost Belyi maps were assertively introduced by Kitaev [19], [20] in the context of algebraic solutions of the Painlevé VI equation.

**Definition 1.1.** An *almost Belyi map* (or an *AB-map*, for shorthand) is an algebraic covering  $\varphi : C \rightarrow \mathbb{P}^1$  that has exactly one simple branching point outside the fibers  $\{0, 1, \infty\} \subset \mathbb{P}^1$ . (Recall that *simple* branching points have the branching order 2.)

Kitaev constructed algebraic Painlevé VI functions using the Jimbo-Miwa correspondence [15] to isomonodromic  $2 \times 2$  Fuchsian systems with 4 singularities. The corresponding Fuchsian systems are generated by pull-backs of the Gauss-Euler hypergeometric equation with respect to AB-maps. In the context of Picard-Fuchs equations, the same pull-back method with AB-maps was employed by Doran [6], Movasatti, Reiter [23].

Recently, algebraic Painlevé VI solutions and AB-maps found application in Saito's singularity theory [25], [26] and Dubrovin's more general theory of Frobenius

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manifolds [7]. This direction [16] motivates computation of new examples of AB-maps. In particular, a list of AB-maps giving pull-backs to all cases of algebraic Painlevé VI solutions in the Lisovsky-Tykhyy classification [22] (up to Schlesinger gauge transformations) is desirable.

The problem of computing AB-maps and the mentioned applications give an interesting set of differential relations for AB-maps and their polynomial components. In particular,

- Usefulness of differentiation in computing Belyi maps was noticed by several authors [27, §2.5]. Further, the very fact of implied pull-back transformations of Fuchsian equations gives additional differential and algebraic restrictions. The same techniques apply to computation of AB-maps, as we demonstrate in §2.3.
- Kitaev’s basic construction entails differentiation with respect to the “isomonodromic” *parameter* (rather than with respect to the independent variable), leading to differential relations between the coefficients of an AB-map. The straightforward case of Kitaev’s *RS-transformations* is summarized in Theorem 2.8.
- Saito’s construction of *free divisors* gives action of vector fields that relates differentiation both with respect to the independent variable and the “isomonodromic” *parameter*. Remarkably, we observe that the action of the vector fields is *logarithmic* even on each polynomial component, leading to Conjecture 3.5.

Analysis of these differential relations (in §2.3, §2.4, §3, respectively) is the main contribution of this article. Additionally, Section 4 presents computational results of AB-maps for all genus 0 and 1 cases of the Lisovsky-Tykhyy classification [22] of algebraic Painlevé VI solutions.

## 2 Preliminaries

Here we introduce application of Belyi maps and AB-maps to pull-back transformations between Fuchsian equations; basic methods for computing these maps and differential relations they employ.

### 2.1 Nomenclature for AB-maps

This paper studies AB-maps of genus 0. We are thus looking at rational functions  $\varphi(x) \in \mathbb{C}(x)$  such that all branching points  $\{x : \varphi'(x) = 0\}$  *except one* lie in the fibers  $\varphi(x) \in \{0, 1, \infty\}$ . The extra branching point has the branching order 2 (thus  $\varphi'' \neq 0$  at that branching point if it is not  $\infty$ ).

Important distinctions between Belyi maps and AB-maps are:

- (i) Belyi maps form discrete (0-dimensional) Hurwitz spaces. AB-maps form 1-dimensional Hurwitz spaces; that is, there are 1-dimensional families of them parametrized by algebraic curves.

- (ii) By Hurwitz theorem, a Belyi map  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  of genus 0, degree  $d$  has exactly  $d + 2$  distinct points in the 3 fibers  $\varphi(x) \in \{0, 1, \infty\}$ . An AB-map  $\varphi : \mathbb{P}^1 \rightarrow \mathbb{P}^1$  of genus 0, degree  $d$  has exactly  $d + 3$  points in the 3 fibers.

**Example 2.1.** An example of a AB-map of degree 6 is

$$\varphi_1(x) = \frac{(w x^3 + 15x^2 + 20x + 8)^2}{64(x + 1)^5}. \quad (1)$$

The parameter  $w$  appears only once. We can compute:

$$\begin{aligned} \varphi_1(x) - 1 &= \frac{x^3(w^2x^3 + 2(15w - 32)x^2 + 5(8w - 19)x + 16w - 40)}{64(x + 1)^5}, \\ \varphi_1'(x) &= \frac{x^2(w x^3 + 15x^2 + 20x + 8)(w x + 6w - 15)}{64(x + 1)^5}. \end{aligned}$$

The root  $x = q_1 = -6 + 15/w$  of  $(w x + 6w - 15)$  is the only branching point outside the fibers  $\varphi(x) \in \{0, 1, \infty\}$ .

**Notation 2.2.** Let  $\varphi \in \mathbb{C}(x)$  be a rational function of degree  $d$ . The *branching pattern* in a fiber  $\varphi = C$  is given by a partition of  $d$ . We choose the multiplicative notation  $1^{n_1}2^{n_2} \dots$  for a branching pattern, meaning  $n_1$  non-branching points,  $n_2$  branching points of order 2, etc. For example, we write the branching pattern of the fiber  $\varphi_1 = 1$  of the AB-map in (1) as  $1^33$  rather than  $1 + 1 + 1 + 3$ . The partition fact is expressed by  $\sum kn_k = d$ .

The collection  $[P_1/P_2/P_3]$  of the branching patterns  $P_1, P_2, P_3$  in the fibers  $\varphi = 0, \varphi = 1, \varphi = \infty$  is called at the *passport* of  $\varphi$ . For example, the passport of  $\varphi_1$  in (1) is  $[2^3/31^3/51]$ , keeping in mind the point  $x = \infty$  in the fiber  $\varphi = \infty$ . The order of branching patterns in the passport is not significant to us, as permutation of the 3 fibers is realized by the fractional-linear expressions  $\varphi/(\varphi - 1), 1 - \varphi, 1/\varphi, 1/(1 - \varphi), (\varphi - 1)/\varphi$ .

## 2.2 Pull-backs of Fuchsian equations

One application of Belyi maps is pull-back transformations of the hypergeometric equation

$$\frac{d^2y(z)}{dz^2} + \left( \frac{c}{z} + \frac{a + b - c + 1}{z - 1} \right) \frac{dy(z)}{dz} + \frac{ab}{z(z - 1)} y(z) = 0. \quad (2)$$

to Fuchsian equations with a few singularities (e.g., Heun, other hypergeometric equations). The pull-back transformations have the form

$$z \mapsto \varphi(x), \quad y(z) \mapsto Y(x) = \theta(x) y(\varphi(x)), \quad (3)$$

where  $\varphi(x)$  is a rational function, and  $\theta(x)$  is a Liouvillian (e.g., power) function. The rational function  $\varphi(x)$  is typically a special Belyi map. Applicable Belyi maps are characterized using the following definition [12, Definition 1.2].

**Definition 2.3.** Given positive integers  $k, \ell, m, n$ , a Belyi map  $\varphi : \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$  is called  $(k, \ell, m)$ -minus- $n$  regular if, with exactly  $n$  exceptions in total, all points above  $z = 1$  have branching order  $k$ , all points above  $z = 0$  have branching order  $\ell$ , and all points above  $z = \infty$  have branching order  $m$ .

The singularities and the local exponents of the pulled-back Fuchsian equation are straightforwardly determined from the pull-back (3) and Riemann's  $P$ -symbol

$$P \left\{ \begin{array}{ccc} 0 & 1 & \infty \\ 0 & 0 & a \\ 1-c & c-a-b & b \end{array} \begin{array}{c} z \\ \\ \end{array} \right\}$$

of hypergeometric equation (2). For the pulled-back Fuchsian equation to have only  $n$  singularities, we usually need the local exponent differences  $c - a - b$ ,  $1 - c$ ,  $b - a$  to be inverse integers  $\pm 1/k$ ,  $\pm 1/\ell$ ,  $\pm 1/m$ , and the covering  $z = \varphi(x)$  to be a  $(k, \ell, m)$ -minus- $n$  regular Belyi map. The canonical Fuchsian equations with  $n \leq 4$  are hypergeometric and Heun equations.

We extend Definition 2.3 to AB-maps.

**Definition 2.4.** Given positive integers  $k, \ell, m, n$ , an AB-map  $\varphi : \mathbb{P}_x^1 \rightarrow \mathbb{P}_z^1$  is called  $(k, \ell, m)$ -minus- $n$  regular if, with exactly  $n$  exceptions in total, all points above  $z = 1$  have branching order  $k$ , all points above  $z = 0$  have branching order  $\ell$ , and all points above  $z = \infty$  have branching order  $m$ .

**Example 2.5.** The AB-map  $\varphi_1(x)$  in Example 2.1 is  $(3, 2, 5)$ -minus-4 regular. The 4 exceptional points are  $x = \infty$  and the 3 simple roots of  $\varphi_1(x) - 1$ .

**Remark 2.6.** Recently, van Hoeij and Kunwar classified  $(2, 3, \infty)$ -minus-5 regular AB-maps in [11]. Here  $\infty$  means that all points in the third fiber are counted as exceptional (towards 5). These maps have degree  $\leq 12$ . A portion of the AB-maps  $N_1, \dots, N_{68}$  in [11, Table 1] are applicable as  $(2, 3, m)$ -minus-4 maps to the Fuchsian equations considered here; see the fifth column in Table 4.1.

Pull-back transformations with respect to  $(k, \ell, m)$ -minus- $n$  regular AB-maps can transform hypergeometric equation (2) with the local exponent differences  $1/k, 1/\ell, 1/m$  to Fuchsian equations with an apparent singularity and  $n$  other singularities. The apparent singular point will have the local exponents 0, 2, rather than 0, 1 for regular points. Since AB-maps are parametrized by algebraic curves, a generic pull-back transformation will give isomonodromic families of Fuchsian equations with these singularities.

An important case is Fuchsian ordinary differential equations with an apparent singularity and  $n = 4$  other singular points. Isomonodromic families of these equations are parametrized by solutions of the Painlevé VI equation

$$\begin{aligned} \frac{d^2 q}{dt^2} = & \frac{1}{2} \left( \frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t} \right) \left( \frac{dq}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t} \right) \frac{dq}{dt} \\ & + \frac{q(q-1)(q-t)}{t^2(t-1)^2} \left( \alpha + \beta \frac{t}{q^2} + \gamma \frac{t-1}{(q-1)^2} + \delta \frac{t(t-1)}{(q-t)^2} \right). \end{aligned} \quad (4)$$

By the Jimbo-Miwa correspondence [15], a solution  $q(t)$  parametrizes isomonodromic  $2 \times 2$  Fuchsian systems  $dY/dx = A(x, t)Y$  with the singularities  $x = 0, x = 1, x = t, x = \infty$  and the local monodromy differences  $\theta_0, \theta_1, \theta_t, \theta_\infty$  such that

$$\alpha = \frac{(\theta_\infty - 1)^2}{2}, \quad \beta = -\frac{\theta_0^2}{2}, \quad \gamma = \frac{\theta_1^2}{2}, \quad \delta = \frac{1 - \theta_t^2}{2}. \quad (5)$$

An equivalent isomonodromic family of ODEs has 1 apparent and 4 other singularities. To write down the parametric Fuchsian ODE explicitly, one can use a specification of the Painlevé VI equation in terms of the Hamiltonian system

$$\frac{dq}{dt} = \frac{\partial \mathcal{H}_0}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}_0}{\partial q}. \quad (6)$$

with

$$\begin{aligned} \mathcal{H}_0 &= \frac{q(q-1)(q-t)}{t(t-1)} \left( p^2 - \left( \frac{\theta_0}{q} + \frac{\theta_1}{q-1} + \frac{\theta_t-1}{q-t} \right) p + \frac{\Theta}{q(q-1)} \right), \\ \Theta &= \frac{(\theta_0 + \theta_1 + \theta_t - \theta_\infty)(\theta_0 + \theta_1 + \theta_t + \theta_\infty - 2)}{4}. \end{aligned} \quad (7)$$

The Painlevé VI equation is obtained by eliminating  $p$ . The corresponding Fuchsian ODE is

$$\begin{aligned} &\frac{d^2 Y(x)}{dx^2} + \left( \frac{1-\theta_0}{x} + \frac{1-\theta_1}{x-1} + \frac{1-\theta_t}{x-t} - \frac{1}{x-q} \right) \frac{dY(x)}{dx} \\ &+ \left( \frac{\Theta}{x(x-1)} + \frac{q(q-1)p}{x(x-1)(x-q)} - \frac{t(t-1)\mathcal{H}_0}{x(x-1)(x-t)} \right) Y(x) = 0; \end{aligned} \quad (8)$$

see [14, pg. 169–173] with  $n = 1$ .

**Notation 2.7.** Let  $P_{VI}(\theta_0, \theta_1, \theta_t, \theta_\infty)$  denote the Painlevé VI equation (5) with the parameters (5). Similarly, let  $E(1-c, c-a-b, b-a)$  denote the hypergeometric equation (2) by the local exponent differences.

### 2.3 Computational methods

As considered in [12, §5.2], a  $(k, \ell, m)$ -minus- $n$  regular Belyi map has the forms

$$\varphi(x) = r_1 \frac{P^\ell F}{Q^m G} \quad (9)$$

$$= 1 + r_2 \frac{R^k H}{Q^m G}, \quad (10)$$

where  $P, Q, R$  are monic polynomials without multiple roots;  $F, G, H$  are monic polynomials with  $n$  or  $n-1$  distinct roots in total; and  $r_1, r_2$  are constants. We refer to the polynomials  $P, Q, R, F, G, H$  as *polynomial components* of  $\varphi$ .

The total number of distinct roots of the 6 polynomial components (including  $x = \infty$  if one of the 3 terms in the polynomial identity is of lower degree) equals

$\deg \varphi + 2$ , by §2.1(ii). The two expressions (9)–(10) are equivalent to the polynomial identity

$$r_1 P^\ell F = Q^m G + r_2 R^k H. \quad (11)$$

A  $(k, \ell, m)$ -minus- $n$  regular *AB-map* has the same shape, but the total number of roots in the terms (including  $x = \infty$ ) equals  $\deg \varphi + 3$  rather than  $\deg \varphi + 2$ . The polynomial components and the constants  $r_1, r_2$  may then depend on a continuous parameter.

Polynomial identity (11) gives a system of necessary polynomial equations for the undetermined coefficients of  $P, Q, R$  and perhaps of  $F, G, H$ . If the degree of the target Belyi map significantly exceeds 10, the algebraic system is too complicated, with too many degenerate (*parasitic*) solutions to be solved by Gröbner basis techniques efficiently. Simpler algebraic systems are obtained by considering the logarithmic derivatives

$$\frac{\varphi'(x)}{\varphi(x)} = \ell \frac{P'}{P} + \frac{F'}{F} - m \frac{Q'}{Q} - \frac{G'}{G}, \quad (12)$$

$$\frac{\varphi'(x)}{\varphi(x) - 1} = k \frac{R'}{R} + \frac{H'}{H} - m \frac{Q'}{Q} - \frac{G'}{G}. \quad (13)$$

The roots of  $\varphi'(x)/\varphi(x)$  are the branching points outside the fibers  $\varphi(x) \in \{0, \infty\}$ , with the multiplicity reduced by 1. This consideration gives the alternative expressions

$$\frac{\varphi'(x)}{\varphi(x)} = h_1 \frac{R^{k-1} H}{P Q S}, \quad \frac{\varphi'(x)}{\varphi(x) - 1} = h_2 \frac{P^{\ell-1} F}{Q R S}. \quad (14)$$

If  $\varphi(x)$  is supposed to be a Belyi map,  $S$  here equals the product of irreducible monic factors of  $F G H$ , each to the power 1. If  $\varphi(x)$  is an AB-map,  $S$  equals this product divided by  $x - q$ , where  $q$  is the (undetermined) extra branching point. If  $x = \infty$  is in the  $\varphi = \infty$  fiber, then  $h_1, h_2$  are equal to the branching order at  $x = \infty$ ; otherwise they are (undetermined) constants. The obtained algebraic system for the coefficients is typically over-determined, with fewer degenerate solutions. According to [5], [27], these differential relations for Belyi maps were noticed by Fricke, Atkin, Swinnerton-Dyer.

Additional algebraic equations are obtained by considering implied pull-back transformations of second order Fuchsian equations. In particular [12, Lemma 5.1], the pull-back

$$z \mapsto \varphi(x), \quad y(z) \mapsto (Q^m G)^a Y(\varphi(x)) \quad (15)$$

transforms the hypergeometric equation (2) with

$$a = \frac{1}{2} \left( 1 - \frac{1}{k} - \frac{1}{\ell} - \frac{1}{m} \right), \quad b = \frac{1}{2} \left( 1 - \frac{1}{k} - \frac{1}{\ell} + \frac{1}{m} \right), \quad c = 1 - \frac{1}{\ell}$$

to the following Fuchsian equation:

$$\begin{aligned} & \frac{d^2 Y(x)}{dx^2} + \left( \frac{S'}{S} - \frac{F'}{\ell F} - \frac{G'}{m G} - \frac{H'}{k H} \right) \frac{dY(x)}{dx} + \\ & + a \left[ b \left( \frac{h_1 h_2 P^{\ell-2} R^{k-2} F H}{Q^2 S^2} - \frac{m^2 Q'^2}{Q^2} - \frac{G'^2}{G^2} \right) + \frac{m Q''}{Q} + \frac{G''}{G} + \right. \\ & \left. + \left( \frac{1}{k} + \frac{1}{\ell} \right) \frac{m Q' G'}{Q G} + \left( \frac{m Q'}{Q} + \frac{G'}{G} \right) \left( \frac{S'}{S} - \frac{F'}{\ell F} - \frac{G'}{m G} - \frac{H'}{k H} \right) \right] Y(x) = 0. \end{aligned} \quad (16)$$

In the context of pull-back transformations to isomonodromic Fuchsian system with one apparent singularity and 4 other singularities, this equation can be compared with (8).

## 2.4 Relation of algebraic Painlevé VI solutions

Kitaev [19], [20] initiated study of AB-maps with the purpose of constructing algebraic Painlevé VI solutions. The relevant AB maps are  $(k, \ell, m)$ -minus-4 regular, as they induce pull-back transformations of to isomonodromic  $2 \times 2$  Fuchsian systems with 4 singularities (or the corresponding ODEs) by the Jimbo-Miwa correspondence [15] of these systems to Painlevé VI solutions. Kitaev's basic construction gives the following result.

**Theorem 2.8.** *Let  $\varphi(X)$  denote a  $(k, \ell, m)$ -minus-4 regular AB-map. Suppose that its irregular branching points are  $X = 0$ ,  $X = 1$ ,  $X = \infty$ ,  $X = t$ . Let  $X = q$  denote the extra branching point of order 2. Then  $q(t)$  is an algebraic Painlevé VI solution with the parameters  $\theta_j = a_j/K_j$  for  $j \in \{0, 1, t\}$ , and  $\theta_\infty = 1 - a_\infty/K_\infty$ . Here  $K_j, K_\infty \in \{k, \ell, m\}$  depending on the fiber of the each of the 4 singularities, and  $a_j, a_\infty$  are the branching orders at them.*

*Proof.* The Jimbo-Miwa correspondence [15] and explicit consideration of a pull-back from  $E(1/\ell, 1/k, 1/m)$ . This is the particular case  $\varepsilon = 1$  of [19, Theorem 2.1].  $\square$

This theorem gives differential relations between *coefficients* of AB-maps. The relation between  $t$  and  $q$  is algebraic because the Hurwitz space is one-dimensional.

**Example 2.9.** Consider the polynomials

$$\begin{aligned} P &= x^4 + 4wx^2 - 6wx + w^2, \\ R &= 2x^6 + 12wx - 18wx^3 + 15w^2x^2 - 36w^2x - w^2(2w - 27), \\ G_1 &= x - 1, \\ G_2 &= 4x^3 + wx^2 + 18wx + w(4w - 27). \end{aligned} \quad (17)$$

Reminiscent to (11), we have a polynomial identity  $4P^3 = R^2 + r_0 G_1^2 G_2$  with  $r_0 = 27w^3$ . It defines a  $(2, 3, 7)$ -minus-4 regular AB-map

$$\varphi_2(x) = \frac{4P^3}{r_0 G_1^2 G_2} = 1 + \frac{R^2}{r_0 G_1^2 G_2} \quad (18)$$

of degree 12, with the branching pattern  $[2^6/3^4/7^2 1^3]$ . The extra branching point is  $x = q_2 = 1 - 2w/7$ . To obtain an algebraic Paineve VI solution of  $P_{VI}(1/7, 1/7, 2/7, 6/7)$  by Theorem 2.8, we first reparametrize

$$w \mapsto -\frac{(s^2 + 3)^3}{(s - 1)^2(s + 1)^2} \quad (19)$$

so that  $G_2$  has rational roots:

$$x_1 = \frac{(s^2 + 3)(s^2 + 15)}{4(s - 1)(s + 1)}, \quad x_2 = \frac{(s^2 + 3)(2s^2 + 3s + 3)}{(1 - s)(s + 1)^2}, \quad x_3 = \frac{(s^2 + 3)(2s^2 - 3s + 3)}{(s - 1)^2(s + 1)}.$$

We move these points to the locations  $X_1 = \infty$ ,  $X_2 = 0$ ,  $X_3 = 1$  by the Möbius  $x$ -transformation

$$x \mapsto \left( \frac{s^2 + 3}{s^2 - 1} \right) \frac{4r^3(s^2 + 15)X - (s - 3)^3(2s^2 + 3s + 3)}{16s^3X + (s + 1)(s - 3)^3}. \quad (20)$$

The root of  $G_1$  is transformed to  $X = t_2$  with

$$t_2 = \frac{(s - 3)^3(s^2 + s + 2)^2}{2s^3(s^2 + 7)^2}, \quad (21)$$

and the transformed location of the extra branching point is

$$q_2 = \frac{(s + 1)(3 - s)(s^2 + s - 2)}{2s(s^2 + 7)}. \quad (22)$$

This parametrizes an algebraic solution  $q_2(t_2)$  of  $P_{VI}(1/7, 1/7, 2/7, 6/7)$ . The fractional-linear transformation  $t_2(q_2 - 1)/(q_2 - t_2)$  permutes the singularities  $0 \leftrightarrow 1$ ,  $t_2 \leftrightarrow \infty$ , and gives the *Kleinian solution* of Boalch [2]. Kitaev derived this solution by the pull-back construction [19, §3.4.3], also followed in [32, §5].

**Example 2.10.** Consider the polynomials

$$\begin{aligned} P &= x^3 + (w - 6)x^2 + 24x - 48, \\ R &= x^5 + 2(w - 6)x + (w^2 - 12w + 72)x^3 + 36(w - 8)x^2 - 72(w - 9)x - 864, \\ F &= x + w - 6, \\ G &= wx^3 + (w^2 - 6w - 3)x^2 + 8(3w + 1)x - 16(4w + 3). \end{aligned} \quad (23)$$

We have a polynomial identity  $P^3 F = R^2 + 1728G$ . It defines a  $(2, 3, 7)$ -minus-4 regular AB-map  $\varphi_3(x)$  of degree 10, with the branching pattern  $[2^5/3^3 1/7 1^3]$ . The extra branching point is  $x = q_3 = -4(w^2 - 6w - 6)/(7w)$ . The curve  $G(x, w) = 0$  defines a genus 0 curve; a parametrization of it gives a substitution after which the polynomial  $G(x)$  has a rational root:

$$w \mapsto \frac{(s + 2)(s^2 + 2s + 9)}{(s - 1)^2}. \quad (24)$$



Complete factorization of  $G$  is achieved on the genus 1 curve  $y^2 = s(s^2 + s + 7)$ . Here are the roots of  $G$ :

$$x_1 = \frac{(1-s)(s+3)}{s+2}, \quad x_2 = \frac{4(2s^2 + 2s + 5 + 3y)}{(s-1)(3-y)}, \quad x_3 = \frac{4(2s^2 + 2s + 5 - 3y)}{(s-1)(3+y)}.$$

The three roots are mapped to  $X_1 = \infty$ ,  $X_2 = 0$ ,  $X_3 = 1$  by the Möbius  $x$ -transformation

$$x \mapsto \frac{4(1-s)(2y(s+3)(s^2+s+7)(2X-1) - 3s^4 - 34s^3 - 114s^2 - 252s - 245)}{8y(s+2)(s^2+s+7)(2X-1) - s^6 - 2s^5 + 9s^4 + 64s^3 + 221s^2 + 210s + 147}.$$

The root of  $F$  is transformed to  $X = t_3$  with

$$t_3 = \frac{1}{2} + \frac{s^9 - 84s^6 - 378s^5 - 1512s^4 - 5208s^3 - 7236s^2 - 8127s - 784}{432(s+1)^2(s^2+s+7)y}, \quad (25)$$

and the transformed location of the extra branching point is

$$q_3 = \frac{1}{2} - \frac{s(s^4 + 2s^3 + 12s^2 + 20s + 73)}{12(s+1)(s+2)y}. \quad (26)$$

This parametrizes an algebraic solution  $q_3(t_3)$  of  $P_{VI}(1/7, 1/7, 1/3, 6/7)$ , of genus 1. An equivalent solution  $t_3(q_3 - 1)/(q_3 - t_3)$  of  $P_{VI}(1/7, 1/7, 1/7, 2/3)$  was first found by Kitaev [20, §3] by the pull-back method.

More generally, Kitaev's method [19] allows further *Schlessinger gauge transformations* to obtain multiple algebraic Painlevé VI solutions from the same pull-back transformation. These transformations are matrix analogues of (3) with  $\varphi(x) = x$ . They shift local exponent differences (including  $\theta_0, \theta_1, \theta_t, \theta_\infty$ ) by integers; the total shift sum must be even. The whole construction is called *RS-transformations*, where R stands for a Rational pull-back, and S stands for a Schlessinger transformation.

**Example 2.11.** Examples 2.9, 2.10 implicitly employ pull-backs of the hypergeometric equation  $E(1/2, 1/3, 1/7)$  to isomonodromic Fuchsian equations with 4 singularities at the roots of  $V_1, V_2$  (or  $U, V$ , respectively) and an apparent singularity at  $x = q_2$  (or  $x = q_3$ ). This lead to algebraic solutions of  $P_{VI}(1/7, 1/7, 2/7, 6/7)$  and  $P_{VI}(1/7, 1/7, 1/3, 6/7)$  by Theorem 2.8. The same pull-back transformations can be applied to the hypergeometric equations  $E(1/2, 1/3, 2/7)$  and  $E(1/2, 1/3, 3/7)$ , as suggested by Kitaev [19], [20]. The pull-backs of  $E(1/2, 1/3, 2/7)$  have the same  $4 + 1$  singularities, plus a new apparent singularity at  $x = \infty$ . Schlesinger transformations neutralizing this singularity give algebraic solutions of  $P_{VI}(2/7, 2/7, 4/7, 2/7)$ ,  $P_{VI}(2/7, 2/7, 1/3, 2/7)$ , as demonstrated in [32]. Similarly, the pull-backs of  $E(1/2, 1/3, 3/7)$  have the same  $4 + 1$  singularities, plus a new singularity at  $x = \infty$  with the monodromy difference 3. Neutralizing Schlesinger transformations lead to algebraic solutions of  $P_{VI}(3/7, 3/7, 6/7, 4/7)$  and  $P_{VI}(3/7, 3/7, 1/3, 4/7)$ , as shown in [32].

It is worth recalling here the *Okamoto* (also called *Backlund*) transformations [24] that convert  $q(t)$  to rational functions of  $q(t)$ ,  $dq/dt$  and  $t$ . The basic transformation acts on the parameters of the Painlevé VI equation as follows:

$$(\theta_0, \theta_1, \theta_t, \theta_\infty) \mapsto (\Theta - \theta_0, \Theta - \theta_1, \Theta - \theta_t, \Theta - \theta_\infty), \quad (27)$$

with  $\Theta = (\theta_0 + \theta_1 + \theta_t + \theta_\infty)/2$ . Special cases are transformations that shift  $(\theta_0, \theta_1, \theta_t, \theta_\infty)$  by integer vectors, with the total shift even. They can be realized by Schlesinger gauge transformations of the Fuchsian equations.

Note that  $P_{VI}(\pm\theta_0, \pm\theta_1, \pm\theta_t, 1 \pm \vartheta_\infty)$  is the same Painlevé VI equation, hence (27) defines 16 “neighbouring” Painlevé VI equations by Okamoto transformations. A set of fractional linear transformations permutes the 4 singular points. All together [24], these transformations form an affine Weil group of type  $E_6$ . Up to the integer shifts and permutation of the singular points, a generic Okamoto orbit contains three distinct Painlevé VI solutions.

**Example 2.12.** The equations

$$P_{VI}(1/7, 1/7, 2/7, 6/7), \quad P_{VI}(2/7, 2/7, 4/7, 2/7), \quad P_{VI}(3/7, 3/7, 6/7, 4/7)$$

in Example 2.11 and their algebraic solutions are related by the Okamoto transformations. But the equations

$$P_{VI}(1/7, 1/7, 1/3, 6/7), \quad P_{VI}(2/7, 2/7, 1/3, 2/7), \quad P_{VI}(3/7, 3/7, 1/3, 4/7)$$

are not related by the Okamoto transformations. For example, the Okamoto orbit of  $P_{VI}(1/7, 1/7, 1/3, 6/7)$  consists of Schlesinger and fractional-linear transformations of itself and  $P_{VI}(17/42, 17/42, 17/42, 5/42)$ ,  $P_{VI}(11/42, 11/42, 11/42, 23/42)$ .

### 3 Differentiation relations from free divisors

Theorem 2.8 gives differential relations between coefficients of AB-maps. Here we observe differential relations with differentiations both with respect to the variable  $x$  and a parameter  $w$ .

#### 3.1 Vector fields, free divisors

As presented in [16], interesting examples of flat structures, free divisors in Saito’s singularity theory [25] can be constructed from algebraic Painlevé VI solutions and corresponding AB-maps. In Dubrovin’s more general context [7] of potentials on Frobenius manifolds, solutions of the Witten-Dijkgraaf-Verlinde-Verlinde equations play a similar key role.

The polynomial components of  $(k, \ell, m)$ -minus-4 regular AB-maps that give the 4 exceptional points typically define *free divisors* after a suitable homogenization.

**Example 3.1.** We homogenize the AB-map  $\varphi_1$  of Example 2.1 by  $w = v/u^3$ ,  $x = uX/v^2$  with the variables  $u, v, X$  of the weighted degrees 1, 3, 5, respectively. The weighted-homogeneous polynomials

$$\begin{aligned} P &= X, & Q &= uX + v^2, \\ R &= X^3 + 15u^2vX^2 + 20uv^3X + 8v^5, \\ F &= X^3 + 2u^2(15v - 32u^3)X^2 + 5uv^2(8v - 19u^3)X + 8(2v - 5u^3)v^4. \end{aligned}$$

Correspondingly, they satisfy  $P^3F + 64Q^5 = R^2$ . Let us consider the differential operators

$$V_1 = u \frac{\partial}{\partial u} + 3v \frac{\partial}{\partial v} + 5X \frac{\partial}{\partial X}, \quad (28)$$

$$V_2 = -2(v - 3u^3) \frac{\partial}{\partial u} + (X + 3u^2v) \frac{\partial}{\partial v}, \quad (29)$$

$$V_3 = 3(X + 27u^2v - 64u^5) \frac{\partial}{\partial u} + 8u(7v - 12u^3)v \frac{\partial}{\partial v} - 40v^3 \frac{\partial}{\partial X}. \quad (30)$$

They are *logarithmic* along the hypersurface  $F = 0$ , as they act on the weighted-homogeneous  $F$  by polynomial multiplication:

$$V_1 F = 15F, \quad V_2 F = 30u^2F, \quad V_3 F = 60(3v - 16u^3)F. \quad (31)$$

Consider the matrix

$$M = \begin{pmatrix} u & 3v & 5X \\ -2(v - 3u^3) & X + 3u^2v & 0 \\ 3(X + 27u^2v - 64u^5) & 8u(7v - 12u^3)v & -40v^3 \end{pmatrix} \quad (32)$$

where the rows represent the vector fields, so that

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = M \begin{pmatrix} \partial/\partial u \\ \partial/\partial v \\ \partial/\partial X \end{pmatrix}.$$

Then  $\det M = -15F$ . Existence of 3 vector fields with the logarithmic action (31) and identification with their determinant up to a constant multiple means that  $F$  is a *free divisor*.

The Euler vector field  $V_1$  acts on the other polynomial components  $P, Q, R$  as multiplication by the weighted-homogeneous degrees 5, 6, 9 (respectively). Remarkably, the vector field  $V_2$  acts on them logarithmically as well:

$$V_2 P = 0, \quad V_2 Q = 6u^2Q, \quad V_2 R = 15u^2R. \quad (33)$$

The isomonodromic Fuchsian system can be elegantly expressed in terms of the vector fields

$$\tilde{V}_2 = V_2 - 2u^2 V_1, \quad \tilde{V}_3 = V_3 + 32u^2 V_2 - 12uv V_1. \quad (34)$$

The action on the AB-map

$$\tilde{\varphi}_1 = -\frac{P^3F}{64Q^5} \quad (35)$$

is

$$V_1 \tilde{\varphi}_1 = 0, \quad \tilde{V}_2 \tilde{\varphi}_1 = 0, \quad \tilde{V}_3 \tilde{\varphi}_1 = -\frac{15R}{PQ} \tilde{\varphi}_1, \quad (36)$$

and the pulled-back hypergeometric function

$$f = Q^{1/12} F^{\lambda/15} {}_2F_1 \left( \begin{matrix} -1/60, 11/60 \\ 2/3 \end{matrix} \middle| \tilde{\varphi}_1 \right) \quad (37)$$

satisfies the differential system

$$\begin{aligned} V_1 f &= (\lambda + \tfrac{1}{2})f, \\ \tilde{V}_2 f &= -\tfrac{1}{2}u^2 f, \\ \tilde{V}_3^2 f &= -((9v + 20u^3)X + 30u^2v^2)f. \end{aligned} \tag{38}$$

The last equation has order 2, just as the hypergeometric equation.

As free divisors and AB-maps are defined for many algebraic Painlevé VI solutions, we checked that attractive differential systems like (38) for pulled-back hypergeometric solutions exists in every computed (and homogenized) case. The computed cases are presented in §4. Existence of vector fields with logarithmic action on the polynomial components as in (34) was observed as well.

**Observation 3.2.** *For every computed AB-map  $\varphi(X:u:v)$  in weighted-homogeneous variables  $u, v, X$  of the respective weights  $N_X, N_u, N_v$ , there is a vector field*

$$\tilde{A}(X, u, v) \frac{\partial}{\partial X} + \tilde{B}(X, u, v) \frac{\partial}{\partial u} + \tilde{C}(X, u, v) \frac{\partial}{\partial v} \tag{39}$$

*linearly independent from the Euler vector field*

$$N_x x \frac{\partial}{\partial x} + N_u u \frac{\partial}{\partial u} + N_v v \frac{\partial}{\partial v} \tag{40}$$

*that acts on all polynomial components of  $\varphi$  logarithmically (that is, by polynomial multiplication).*

This observation is remarkable. As a weaker implication, it says that there are low degree syzygies between  $\partial W/\partial X$ ,  $\partial W/\partial u$ ,  $\partial W/\partial v$  and  $W$  for any polynomial component  $W \in \{P, Q, R, F, G, H\}$  as in (9)–(10). We found the remarkable vector fields by computing the lowest degree syzygy between the derivatives of  $R$ , and checking the observation on other polynomial components. The chosen syzygy is always much smaller than alternatives.

### 3.2 Dehomogenization

Observation 3.2 can be modified to apply to non-homogeneous AB-maps  $\varphi(x, w)$ , claiming a single vector field that acts logarithmically on all polynomial components of  $\varphi$ . If de-homogenization of  $\varphi(X:u:v)$  is simply  $u = 1$ , one can find a linear combination of (39) and (40) with eliminated  $\partial/\partial u$ , and then specialize to  $u = 1$ .

**Example 3.3.** Recall Example 2.1 and consider the vector field

$$\mathcal{L}_1 = -2x(x+1) \frac{\partial}{\partial x} + (wx + 6w - 15) \frac{\partial}{\partial w}. \tag{41}$$

This vector field acts on all polynomial components of  $\varphi_1(x)$  logarithmically, including on  $x$  and  $x+1$ . To derive this vector field from Example 3.1, we substitute

$$\begin{aligned} \frac{\partial}{\partial u} &= \frac{x}{v^2} \frac{\partial}{\partial x} - \frac{3v}{u^4} \frac{\partial}{\partial w} = \frac{1}{u} \left( x \frac{\partial}{\partial x} - 3w \frac{\partial}{\partial w} \right), \\ \frac{\partial}{\partial v} &= -\frac{2uX}{v^3} \frac{\partial}{\partial x} + \frac{1}{u^3} \frac{\partial}{\partial w} = \frac{1}{v} \left( -2x \frac{\partial}{\partial x} + w \frac{\partial}{\partial w} \right) \end{aligned}$$

into  $V_2$ , and recognize  $\mathcal{L}_1$  after multiplication by  $u/v$ .

**Example 3.4.** For Example 2.9, the vector field

$$\mathcal{L}_2 = (x-1)(3x+w)\frac{\partial}{\partial x} + w(7x+2w-9)\frac{\partial}{\partial w} \quad (42)$$

acts on  $P, R, G_1, G_2$  and even on  $r_0$  logarithmically. Considering  $\varphi_2(x, s)$  after the substitution (19), the vector field

$$\tilde{\mathcal{L}}_2 = -14(s^2+3)x(x-1)\frac{\partial}{\partial x} + (2s(s^2+7)x + (s+1)(s-3)(s^2+s+2))\frac{\partial}{\partial s}$$

acts on the polynomial components (with cleared denominators  $\in \mathbb{Q}[s]$ ) logarithmically as well. As  $G_2$  factors  $(x-x_1)(x-x_2)(x-x_3)$  over  $\mathbb{Q}(s)$ , the vector field acts logarithmically even on all three factors (with cleared denominators).

For Example 2.10, the vector field

$$\mathcal{L}_3 = (x^2 - 2(w+3)x + 24)\frac{\partial}{\partial x} + (7wx + 4w^2 - 24w - 24)\frac{\partial}{\partial w} \quad (43)$$

acts on  $P, R, F, G$  logarithmically.

Notice incidentally, that the coefficient to  $\partial/\partial u$  is linear in  $x$  in these examples, and its root is the extra branching point  $q_j$  with  $j \in \{1, 2, 3\}$ . This follows from another observation that these operators  $\mathcal{L}_j$  annihilate the respective AB-maps  $\varphi_j$ , as the extra branching point is the only root of  $\partial\varphi_j/\partial x$  that is not a root of  $\partial\varphi_j/\partial w$ .

Observation 3.2 becomes simpler in a dehomogenized form. We formulate the following conjecture.

**Conjecture 3.5.** *For any AB-map  $\varphi(x, w)$  with a definition field  $K = \mathbb{Q}(w)$ , the vector field*

$$A(x, w)\frac{\partial}{\partial x} + B(x, w)\frac{\partial}{\partial w} \quad (44)$$

*annihilating  $\varphi(x, w)$  acts on all polynomial components of  $\varphi$  logarithmically. Here by a polynomial component here we mean any  $K[x]$ -irreducible factor of the numerators and the denominators of  $\varphi$  and  $\varphi - 1$ .*

As exemplified above, the conjecture implies that  $B(x, w)$  is linear in  $x$ , and its root gives the extra branching point of  $\varphi$  outside the critical fibers  $\{0, 1, \infty\}$ . By the asymptotics at  $x = \infty$ , the degree of  $A(x, w)$  in  $x$  is at most 2.

We checked the conjecture for all known AB-maps, including the  $(2, 3, \infty)$ -minus-5 maps from [16] that we mentioned in Remark 2.6. Explicit prior knowledge of these vector fields should be very useful in speeding up computation of a desired AB-map, by utilizing new algebraic equations for undetermined coefficients.

## 4 Algebraic Painlevé VI solutions

Algebraic solutions of the Painlevé VI equation were recently classified by Lisovsky and Tykhyy [22]. Apart from rational and an infinite family of Picard, Hitchin's solutions [10] there is a finite list (up to Okamoto transformations) of 3 parametric and 45 non-parametric solutions. The non-parametric solutions were already derived by Dubrovin, Mazzocco [8], Kitaev [19], [20] and Boalch [2], [3], [4] in 2000–2007.

#### 4.1 AB-maps for algebraic Painlevé VI solutions

Kitaev conjectured [18] that all algebraic solutions of the Painlevé VI equation can be obtained from pull-back transformations by  $(k, \ell, m)$ -minus-4 regular AB-maps, up to Okamoto and Schlesinger transformations. By checking the Lisovsky-Tykhyy classification we see that this conjecture is true for the  $3 + 45$  solutions in [22]:

- (i) The 3 Okamoto orbits of parametric solutions have corresponding pull-back transformations, as established by Kitaev [19].
- (ii) The Lisovsky-Tykhyy solutions #8, #33 are obtained by the pull-back maps  $\varphi_2(x)$ ,  $\varphi_3(x)$  of Examples 2.9, 2.10. The similar solutions #32, #34 solve  $P_{VI}(2/7, 2/7, 1/3, 2/7)$  and  $P_{VI}(3/7, 3/7, 1/3, 4/7)$ . They are obtained from  $\varphi_3(x)$  by additional Schlesinger transformations described in Example 2.11.
- (iii) The other solutions in [22] correspond (up to Okamoto transformations) to isomonodromic Fuchsian equations with finite monodromy. Existence of pull-backs is implied by celebrated Klein's theorem [21]: any second order Fuchsian equations with finite monodromy is a pull-back of a hypergeometric equation with finite monodromy.

In (iii), there are 33 Okamoto orbits corresponding to Fuchsian systems with the icosahedral monodromy group; and 7 octahedral (#4, #5, #9, #10, #20, #21, #30), 1 tetrahedral (#3) cases. As Schlesinger transformations do not change monodromy of Fuchsian equations, the exponent differences  $\theta_0, \theta_1, \theta_t, \theta_\infty$  can be shifted by integers. This gives infinitely many Kleinian pull-backs by AB-maps of unbounded degree in these Okamoto orbits. Okamoto transformations are necessary, as (for example, #16, #17, #31 in [22]) the Dubrovin-Mazzocco solutions of  $P_{VI}(0, 0, 0, 4/5)$ ,  $P_{VI}(0, 0, 0, 2/5)$ ,  $P_{VI}(0, 0, 0, 2/3)$  correspond to Fuchsian systems with logarithmic singularities and cannot be obtained directly by a pull-back transformation.

With the construction of Theorem 2.8 in mind, we computed AB-maps for all Lisovsky-Tykhyy cases algebraic Painlevé VI solutions of genus 0 or 1. The results are presented in Table 4.1, with the genus 0 and 1 cases separated by a horizontal line. The first column gives the enumeration in [22].

The second column of Table 4.1 gives the branches permutation monodromy of the Painlevé VI solutions, using the fact that in a parametrization  $(q(s), t(s))$  of those algebraic solutions,  $t(s)$  is a Belyi map (by the Painlevé property). The second column gives the passport of that Belyi map (without the  $[]$  delimiters), but the notation is compacted when branching patterns in 2 or all 3 fibers is the same. The repetition is indicated by the number of  $'$ 's. For example,  $3^2//2^21^2$  for the solution #3 means the passport  $[3^2/3^2/2^21^2]$ , and  $3^22^2///$  for the solution #15 means the passport  $[3^22^2/3^22^2/3^22^2]$ , etc. The algebraic degree of the Painlevé VI solution can be quickly determined from the passport.

The third column gives the exponent differences of representative Painlevé VI equations  $P_{VI}(\theta_0, \theta_1, \theta_t, \theta_\infty)$ . Two distinct Painlevé VI equations are given for the parametric solution IV, because they are generally not related by Schlesinger and fractional-linear transformations, and AB-maps (of degree 3 and 6) exist for both of them. The case #30 is represented by two Painlevé VI equations solutions for the

Painlevé VI solution			Almost Belyi map		Braid monodromy	
[22]	Monodromy	Exp. differences	Passport	Ref. or $d$	Passport	$d^*$
II	$2//1^2$	$a, a, b, 1 - b$	$1^2/1^2/2$	$N_1/N_2$	$1/1/1$	1
III	$21//3$	$a, a, 2a, 2/3$	$1^2 2/31/2^2$	$[19], N_7/N_9$	$3/21/21$	3
IV	$31///$	$a, a, a, 1/2$	$1^3/21/3$	$[19], N_3/N_4$	$1/1/1$	1
	$(a=2b \pm \frac{1}{2})$	$b, b, b, 1 - 3b$	$31^3/3^2/2^3$	$[19], N_{23}$	$2/2/1^2$	2
1	$32//2^21$	$1/3, 1/3, 1/5, 3/5$	$3^2 1^2/512/2^4$	$N_{33}$	$73/4321/2^4 1^2$	10
2	$32//31^2$	$1/5, 1/5, 2/5, 2/5$	$51^2 23/3^4/2^6$	$[19], N_{61}$	$654/3^4 21/2^7 1$	15
3	$3^2//2^2 1^2$	$1/3, 1/3, 1/2, 1/2$	$31/31/21^2$	$[1], N_6$	$42/42/31^3$	6
		$1/2, 1/2, 1/3, 1/3$	$2^2 1^2/312/3^2$	$N_{19}$	$532^2/3^3 21/3^3 21$	12
4	$42/3^2/2^2 1^2$	$1/4, 1/2, 1/3, 1/2$	$412/3^2 1/2^3 1$	$N_{28}$	$6^2 53^2 1/4^2 3^4 2^2/3^3 2^6 1^3$	24
5	$42//31^3$	$1/4, 1/4, 1/3, 1/3$	$41^2/312/2^3$	$[19], N_{24}$	$531/531/2^4 1$	9
		$1/3, 1/3, 1/4, 1/4$	$3^2 1^2/413/2^4$	$N_{34}$	$7431/4^2 3^2 1/2^7 1$	15
6	$321//51$	$1/5, 2/5, 2/5, 2/3$	$512^2/3^3 1/2^5$	$N_{52}$	$732^2/43^3 2/2^7 1$	15
7	idem	$1/3, 1/5, 1/5, 2/5$	$3^3 1/51^2 3/2^5$	$N_{50}$	$8421/43^3 2/2^7 1$	15
8	$32^2///$	$1/7, 1/7, 1/7, 5/7$	$71^3 2/3^4/2^6$	$[19], N_{57}$	$43/3^2 1/2^3 1$	7
9	$3^2 2//3^2 1^2$	$1/4, 1/4, 1/2, 1/2$	$41^2/2^2 1^2/3^2$	$[19], N_{18}$	$51/321/321$	6
10	$42^2//32^2 1$	$1/4, 1/3, 1/3, 1/2$	$41/31^2/2^2 1$	$N_{13}$	$532/4^2 1^2/3^2 21^2$	10
11	$32^2 1//53$	$1/5, 1/5, 2/5, 1/2$	$51^2 2/2^4 1/3^3$	$N_{43}$	$7632/3^5 21/3^2 2^5 1^2$	18
12	idem	$1/5, 2/5, 2/5, 1/2$	$5^2 12^2/2^7 1/3^5$	$d = 15$	$7^3 6531/3^{11} 21/3^2 2^{14} 1^2$	36
15	$3^2 2^2///$	$1/5, 2/5, 1/2, 1/2$	$5^3 12/2^8 1^2/3^6$	$d = 18$	$7^2 65^7 32/3^{18} 2^3/3^5 2^{21} 1^3$	60
16	$531^2///$	$2/5, 2/5, 2/5, 2/5$	$5^3 2^3 3/3^8/2^{12}$	$d = 24$	$75^2 3/3^6 1^2/2^{10}$	20
17	idem	$1/5, 1/5, 1/5, 1/5$	$51^3 4/3^4/2^6$	$[19], N_{59}$	$532/3^3 1/2^5$	10
18	$52^2 1///$	$1/3, 1/3, 1/3, 4/5$	$31^3/51/2^3$	$N_{21}$	$41/32/2^2 1$	5
19	idem	$1/3, 1/3, 1/3, 2/5$	$3^5 1^3/5^3 3/2^9$	$d = 18$	$852/43^3 1^2/2^7 1$	15
21	$4^2 2^2//3^2 2^2 1^2$	$1/3, 1/3, 1/2, 1/2$	$3^2 1^2/2^3 1^2/4^2$	$d = 12$	$4^2 31/4^2 31/3^2 2^2 1^2$	12
25	$532^2//3^2 2^2 1^2$	$1/3, 1/5, 2/5, 1/2$	$3^4 1/5^2 12/2^6 1$	$d = 13$	$7^4 6^3 5^6 431/4^4 3^{20} 2^4/3^6 2^{30} 1^6$	84
30	$3^4 2^2///$	$1/4, 1/2, 1/2, 1/2$	$4^2 1/2^3 1^3/3^3$	$d = 9$	$54^2 21/3^4 2^2/3^4 21^2$	16
		$1/8, 1/8, 1/8, 7/8$	$81^4/3^4/2^6$	$[20], N_{56}$	$31/31/2^2$	4
13	$5311///$	$2/5, 2/5, 2/5, 2/3$	$5^2 2^3/3^5 1/2^8$	$d = 16$	$7322/3^3 21/2^6$	12
14	idem	$1/5, 1/5, 1/5, 1/3$	$51^3/3^2 2/2^4$	$[19], N_{37}$	$51/321/2^3$	6
20	$4^2 2^2//3^4$	$1/2, 1/3, 1/2, 1/2$	$4^2 2/3^3 1/2^4 1^2$	$d = 10$	$6^3 4^3 3^2/4^2 3^7 2^2 1^3/3^6 2^8 1^2$	36
22	$5^2 2//3^2 2^2 1^2$	$1/3, 1/3, 1/5, 2/5$	$3^4 1^2/5^2 13/2^7$	$d = 14$	$8^3 6^2 5421/4^4 3^{10} 1^2/2^3 1^2$	48
23	$532^2//53^2 1$	$1/5, 1/5, 1/3, 1/2$	$51^2/3^2 1/2^3 1$	$N_{27}$	$6^3 21/4^2 3^3 21^2/3^3 2^5 1^2$	21
24	idem	$2/5, 2/5, 1/3, 1/2$	$5^3 2^2/3^6 1/2^9 1$	$d = 19$	$7^4 5^4 432/4^2 3^{15} 21^2/3^3 2^{23} 1^2$	57
26	$53^2 2^2///$	$1/3, 1/3, 1/3, 3/5$	$3^3 1^3/5^2 2/2^6$	$[31]$	$753/4^2 321^2/2^7 1$	15
27	idem	$1/3, 1/3, 1/3, 1/5$	$3^7 1^3/5^4 4/2^{12}$	$d = 24$	$9^2 5^5 2/4^3 3^{10} 1^3/2^{22} 1$	45
28	$5^2 32//53^2 1^4$	$1/3, 1/3, 2/5, 2/5$	$3^6 1^2/5^3 23/2^{10}$	$d = 20$	$8^2 765^7 4^2 3/4^5 3^{17} 21^2/2^{37} 1$	75
29	idem	$1/5, 1/5, 1/3, 1/3$	$5^2 1^2/3^3 12/2^6$	$d = 12$	$6^4 5^3 42/5^3 3^8 2^2 1^2/2^{22} 1$	45
31	$5^2 3^2 1^2///$	$1/3, 1/3, 1/3, 1/3$	$3^5 1^3 2/5^4/2^{10}$	$[31]$	$5^5 32/54^2 3^5 1^2/2^{15}$	30
33	$73^2 2^2 1///$	$1/7, 1/7, 1/7, 2/3$	$71^3/3^3 1/2^5$	$[20], N_{48}$	$861/43^3 1^2/2^7 1$	15
35	$5^2 3^2 2^2//5^2 3^2 1^4$	$2/5, 2/5, 1/2, 1/2$	$5^4 2^2/2^{11} 1^2/3^8$	$d = 24$	$7^4 5^5 43/3^{19} 21/3^3 2^{24} 1^3$	60
36	idem	$1/5, 1/5, 1/2, 1/2$	$5^2 1^2/2^5 1^2/3^4$	$d = 12$	$6^3 5^2 2/3^9 21/3^3 2^9 1^3$	30
37	$5^2 3^2 2^2//53^2 2^4 1$	$2/5, 1/3, 1/3, 1/2$	$5^3 2/3^5 1^2/2^8 1$	$d = 17$	$7^4 65^9 42/4^6 3^{18} 2^2 1^3/3^5 2^{33} 1^4$	85
38	idem	$1/5, 1/3, 1/3, 1/2$	$5^2 1/3^3 1^2/2^5 1$	$[31]$	$6^3 5^6 43/4^6 3^8 2^2 1^3/3^5 2^{18} 1^4$	55
39	$5^2 3^2 2^4///$	$1/3, 1/3, 1/3, 1/2$	$3^4 1^3/2^7 1/5^3$	$d = 15$	$5^5 32/4^4 3^4 1^2/3^2 2^{11} 1^2$	30

Table 1: AB-maps for algebraic Painlevé VI solutions of genus 0 (in the upper part) or genus 1 (in the lower part).

same reason, while #3, #5 take two lines each because two AB-maps for them are already known.

The fourth column gives the passport of a  $(2, 3, m)$ -minus-4 regular AB-map giving an algebraic solution of  $P_{VI}(\theta_0, \theta_1, \theta_t, \theta_\infty)$  by Theorem 2.8. The three fibers are ordered to match the order of the  $\theta_j$ 's in the third column conveniently. The fifth column either gives the degree  $d$  of the AB-map if it was not computed previously, or gives references to [11, Table 1] (by the  $N_j$ -label) and other publications [1], [19], [20], [32]. Given  $\theta_0 > 0, \theta_1 > 0, \theta_t > 0, \theta_\infty < 1$ , the degree of the pull-back map from  $E(1/2, 1/3, 1/m)$  equals

$$d = \frac{\theta_0 + \theta_1 + \theta_t - \theta_\infty}{\frac{1}{2} + \frac{1}{3} + \frac{1}{m} - 1}. \quad (45)$$

This follows from the Hurwitz theorem, or (assuming the AB-map is defined over  $\mathbb{R}$ ) by geometric consideration of spherical or hyperbolic areas in analytic continuation of pulled-back hypergeometric functions by the Schwarz reflection principle [32, Lemma 6.2, etc.].

The last two columns characterize an important Belyi map derived from each AB-map  $\varphi(x, w)$ . All presented AB-maps are parametrized (as Hurwitz spaces of dimension 1) by algebraic curves of genus 0, with  $w$  as a minimal projective parameter of those curves. The fourth fiber  $\psi(w) = \varphi(q, w)$  of the extra branching point  $x = q$  is a function of  $w$  that is intrinsic to  $\varphi(x, w)$ . It gives the braid group action on  $\varphi(x, w)$  as the fourth fiber is moved continuously around the other three fibers. The function  $\psi(w)$  is a Belyi map [11, Remark 5.3], and is a good measure of complexity of the AB-map. The passport and degree  $d^*$  of  $\psi(w)$  are given in the last two columns of Table 4.1. For the  $w$ -values in the three critical fibers of  $\psi(w) \in \{0, 1, \infty\}$ , the AB-map specializes to Belyi maps of degree  $\leq d$ .

**Remark 4.1.** The cases #32, #34 are skipped in Table 4.1, because Schlesinger transformations are necessary to obtain those Painlevé VI solutions. As we discussed in Example 2.11, the AB-map of #33 has to be applied for a pull-back from  $E(1/2, 1/3, 2/7)$  or  $E(1/2, 1/3, 3/7)$ . Kitaev [19] stresses that pairs of icosahedral cases with the same monodromy (such as #6, #7; see the second column in Table 4.1) can be similarly obtained by pull-backs with respect to a common AB-map applied to  $E(1/2, 1/3, 1/5)$  and  $E(1/2, 1/3, 2/5)$ , with a Schlesinger transformation necessary after one or other pull-back.

Examples of AB-maps for the solutions #40 – #45 in [22] of genus 2, 3 or 7 remain to be computed. But even these cases can be considered as handled if we allow Kitaev's quadratic transformations [17] of Painlevé VI solutions and corresponding isomonodromic Fuchsian systems. Derivation of the Painlevé VI solutions by these quadratic transformations is demonstrated in [30].

The AB-maps presented in Table 4.1 are not necessarily unique for the passports given in fourth column. For example, [11, Table 1] gives also composite maps with the degree 6 and 12 passports of the entries #9 and #30. Another composite map with the degree 20 passport for #31 is given in [32, §5]. As explained in [19], compositions of Belyi maps with an AB-map  $\varphi_0$  give Painlevé VI solutions (by the RS-transformations) that can be obtained from  $\varphi_0$  already. Thus composite AB-maps are not useful in deriving complicated algebraic Painlevé VI solutions.



## 4.2 Computation of AB-maps

Here we demonstrate computation of AB-maps for the Painlevé VI solutions #15 and #22. As these examples show, identification of Fuchsian equations (8), (15) using Painlevé VI solutions straightforwardly gives the singularity polynomials  $F, G, H$ ,  $x - q$  of the AB-maps (and accessory parameters of Fuchsian equations) and a ready, convenient parameter of the Hurwitz curve.

**Example 4.2.** To find an AB-map with the passport  $[3^6/5^3 2^1/2^8 1^2]$  for the algebraic solution #15, we are looking for a polynomial identity

$$P^3 + r_0 Q^5 G = R^2 H \quad (46)$$

with  $P = x^6 + a_1 x^5 + \dots + a_6$ ,  $Q = x^3 + b_1 x^2 + b_2 x + b_3$ ,  $R = x^8 + c_1 x^7 + \dots + c_8$ ,  $G = x$  and  $H = x^2 + d_1 x + d_2$ . After clearing denominators in the logarithmic derivative ansatz (12)–(14) with  $h_1 = h_2 = 2$ ,  $S = GH/(x - q)$ , we get the equations

$$\begin{aligned} 0 &= (2q + 7b_1 - a_1 - 2c_1) x^8 + (12b_2 - 4a_2 - 2c_2 + 2qc_1 + 4a_1 b_1) x^7 + \dots \\ 0 &= (2q + 7b_1 - 4a_1 + d_1) x^{12} + (12b_2 - 4a_2 - 2c_2 + 4qa_1 + 5b_1 c_1 + \dots) x^{11} + \dots \end{aligned}$$

From their leading coefficients we can consequently eliminate all coefficients of  $P$ ,  $R$  except  $a_2$ . Next we compute the pull-back (15)–(16), with  $k = 3$ ,  $\ell = 2$ ,  $m = 5$ , thus  $a = -1/60$ ,  $b = 11/60$ . The coefficient to  $Y(x)$  in (15) equals

$$U_1 = \frac{27x^9 + (11a_1 + 82b_1 - 104d_1 - 289q)x^8 + (11a_2 + 282b_2 - 224d_2 - \frac{11}{4}b_1^2 + \dots)x^7 + \dots}{900(q - x) H G^2 Q^2}.$$

To compute the corresponding equation (8), we start with this solution  $q_{15}(t_{15})$  of  $P_{VI}(1/5, 1/2, 1/2, 3/5)$ :

$$q_{15} = -\frac{2s(s-1)(s-5)^2(s^2-3)(s^2+4s+5)}{(s+1)^2(s+5)(s^2-4s+5)(s^4+6s^2-75)}, \quad (47)$$

$$t_{15} = -\frac{(s-1)^3(s-5)^3(s^2+4s+5)^2}{(s+1)^3(s+5)^3(s^2-4s+5)^2}. \quad (48)$$

It differs from the solution of  $P_{VI}(1/2, 1/5, 1/2, 2/5)$  in [22] by the fractional-linear transformation  $(q_{15}, t_{15}) \mapsto (1 - q_{15}, 1 - t_{15})$ . We express the entities in (6)–(8) in the parametrized form:

$$p_{15} = -\frac{s(s+1)^2(s+5)(s^2-4s+5)(s^4+6s^2-75)}{10(s-1)(s-5)^2(s^4-25)(s^2+4s+5)}, \quad \Theta = -\frac{3}{100}, \quad \text{etc.}$$

The symmetry between  $x = 1$  and  $x = t_{15}$  is realized by  $s \mapsto -s$ . To identify  $(x-1)(x-t_{15})$  with the irreducible polynomial  $H$ , we scale  $x \mapsto x/K$  with

$$K = s(s+1)^3(s+5)^3(s^2-4s+5)^2.$$

The coefficient to  $Y(x)$  in (8) is thereby divided by  $K^2$  (along with the substitution of  $x$ ) and becomes a function of the invariant  $u = s^2$ :

$$U_2 = \frac{3x^2 + \frac{6u(41u^6-900u^5+\dots+46875)}{u^2+6u-75}x + \frac{4u^2(u-1)(u-3)(u-25)^2(u^2-6u+25)(5u^5+\dots-9375)}{u^2+6u-75}}{100(q-x)GH},$$

with explicitly

$$\begin{aligned} H &= x^2 - 4u(5u^4 - 80u^3 + 678u^2 - 2000u + 3125)x - u(u-1)^3(u-25)^3(u^2 - 6u + 25)^2, \\ q &= -\frac{2u(u-1)(u-3)(u-25)^2(u^2 - 6u + 25)}{u^2 + 6u - 75}. \end{aligned} \quad (49)$$

This parametrizes  $d_1, d_2, q$ . The remaining coefficients  $a_2, b_1, b_2, b_3$  are obtained from the identification  $U_1 = U_2$ . After clearing denominators, we get a polynomial expression of degree 8 in  $x$ . The leading coefficients gives immediately

$$b_1 = -\frac{8u(u^6 - 15u^5 - 14u^4 + 3326u^3 - 29575u^2 + 100625u - 187500)}{u^2 + 6u - 75}. \quad (50)$$

The coefficient to  $x^7$  is linear in  $a_2, b_2$ , and the next two coefficients are linear in  $b_3$ . After elimination of  $b_2, b_3$ , we get a quadratic polynomial in  $a_2$  that factorizes. We check both candidates for  $a_2$  on another equation, and the correct value is

$$\begin{aligned} a_2 &= -4u(u^{10} + 1340u^8 - 38600u^7 + 421150u^6 - 3081320u^5 + 20032500u^4 \\ &\quad - 97975000u^3 + 131015625u^2 + 703125000u - 2109375000). \end{aligned}$$

This gives

$$\begin{aligned} b_2 &= -\frac{64u(u-25)^2(11u^6 - 165u^5 + 968u^4 - 3082u^3 + 6875u^2 - 20625u + 31250)}{u^2 + 6u - 75}, \\ b_3 &= \frac{512u^2(u-3)(u-25)^6(u^2 - 6u + 25)^2}{u^2 + 6u - 75} \end{aligned}$$

and the other coefficients. The factor  $r_0$  can be determined by dividing the left-hand side of (46) by  $H$  with respect to  $x$ , and looking at the remainder. We find  $r_0 = 27u(u^2 + 6u - 75)^5$ .

Simplification of the obtained AB-map to a presentable size is a tedious, less automated task that may take much more time than the above computation. The basic ideas are to simplify the Belyi map  $\varphi(q(u), u)$  stated in the last two columns in Table 4.1 (of degree 60); simplification of elliptic surfaces such as  $y^2 = GQ$ ; and considering factorization of the discriminants, resultants of  $P, Q, R, H$  with respect to  $x$ . For example, the transformation  $u = 5v$ ,  $x = 100x + 500v(v-5)^2(5v^2 - 6v + 5)$  is useful for a start, introducing high powers of  $(v-1)$  in the coefficients while keeping the powers of  $v, v-5, 5v^2 - 6v + 5$ .

**Example 4.3.** To find an AB-map with the passport  $[3^4 1^2 / 5^2 1^3 / 2^7]$  for the Painlevé VI solution #22, we are looking for a polynomial identity

$$P^3 F + r_0 Q^5 G = R^2 \quad (51)$$

with  $P = x^4 + a_1 x^3 + \dots + a_4$ ,  $Q = x^2 + b_1 x + b_2$ ,  $R = x^7 + c_1 x^6 + \dots + c_7$ ,  $F = x^2 + d_1 x + d_2$  and  $G = x + e_1$ . We do not hurry with setting  $e_1 = 0$  by choosing the point  $x = 0$ . In the logarithmic derivative ansatz we have  $h_1 = h_2 = 3$ ,  $S = FG/(x - q)$ . It allows to eliminate straightforwardly all coefficients of  $P, R$  except  $a_3$ . We compute the coefficient to  $Y(x)$  in (15), and denote it by  $U_1$  analogously to (4.2).

To compute the Fuchsian equation (8), we use the Painlevé VI solution of  $P_{VI}(1/3, 1/3, 1/5, 2/5)$  from [22], with  $z = \sqrt{3(5s+1)(8s^2-9s+3)}$ :

$$q_{22} = \frac{1}{2} + \frac{140s^6 + 1029s^5 - 1023s^4 + 360s^3 - 288s^2 + 27s + 27}{18z(s+1)(7s^3 - 3s^2 - s + 1)}, \quad (52)$$

$$t_{22} = \frac{1}{2} + \frac{40s^6 + 540s^5 - 765s^4 + 540s^3 - 270s^2 + 27}{6z(s+1)^2(8s^2 - 9s + 3)}. \quad (53)$$

We wish to utilize the symmetry  $z \mapsto -z$ ,  $(q_{22}, t_{22}) \mapsto (1 - q_{22}, 1 - t_{22})$  while identifying  $FG$  with  $x(x-1)(x-t_{22})$ . For this purpose we find an elliptic surface that is defined over  $\mathbb{Q}(t(1-t))$  and has the same  $j$ -invariant as the Legendre family  $y^2 = x(x-1)(x-t)$ . The following elliptic surface has these properties:

$$y^2 = (x-t)(x-1+t)(x-2t(1-t)). \quad (54)$$

Therefore we identify

$$\begin{aligned} F &= (x-t_{22})(x-1+t_{22}) = x^2 - x + t_{22}(1-t_{22}), \\ G &= x - 2t_{22}(1-t_{22}) \end{aligned} \quad (55)$$

initially. Here  $t_{22}(1-t_{22})$  is not dependent on  $z$ :

$$t_{22}(1-t_{22}) = -\frac{16s^5(s-3)^5(5s-3)^2}{27(s+1)^4(5s+1)(8s^2-9s+3)^3}. \quad (56)$$

Additionally, we transform

$$x \mapsto \frac{1}{2} + \frac{40s^6 + 540s^5 - 765s^4 + 540s^3 - 270s^2 + 27}{54(s+1)^4(5s+1)(8s^2-9s+3)^3} x \quad (57)$$

to get the simpler

$$\begin{aligned} F &= x^2 - 27(s+1)^4(5s+1)(8s^2-9s+3)^3, \\ G &= x + 40s^6 + 540s^5 - 765s^4 + 540s^3 - 270s^2 + 27. \end{aligned} \quad (58)$$

This parametrizes  $d_1, d_2, e_1$ . An isomorphism from the Legendre curve to (54) is given by  $x \mapsto (x-t)/(1-2t)$ . The composition of this isomorphism (with  $t = t_{22}$ ) and (57) is the transformation

$$x \mapsto Kx + \frac{1}{2}, \quad \text{with} \quad K = -\frac{z}{18(s+1)^2(5s+1)(8s^2-9s+3)^2}. \quad (59)$$

After this whole transformation, the coefficient to  $Y(x)$  in (8) equals

$$U_2 = \frac{77x^2 + \frac{8(30625s^9 + \dots - 2673s + 2673)}{3(7s^3 - 3s^2 - s + 1)}x + \frac{(s+1)(8s^2-9s+3)(666400s^{12} + \dots + 136323)}{3(7s^3 - 3s^2 - s + 1)}}{900(q-x)FG}$$

with

$$q = -\frac{(s+1)(8s^2-9s+3)(140s^6 + 1029s^5 - 1023s^4 + 360s^3 - 288s^2 + 27s + 27)}{3(7s^3 - 3s^2 - s + 1)}.$$

The identification  $U_1 = U_2$  leads to a polynomial of degree 6 in  $x$  after clearing the denominators. Its 3 leading coefficients give straightforwardly

$$\begin{aligned} b_1 &= \frac{2(8s^2 - 9s + 3)^2(16s^4 - 8s^3 + 8s^2 + 15s + 3)}{7s^3 - 3s^2 - s + 1}, \\ b_2 &= - \frac{(s + 1)^2(8s^2 - 9s + 3)^3(625s^6 + 1386s^5 - 567s^4 + 540s^3 - 27s^2 - 162s - 27)}{7s^3 - 3s^2 - s + 1}, \\ a_3 &= -2(8s^2 - 9s + 3)^3(192500s^{10} + 300697s^9 + 68513s^8 + 41532s^7 + 297588s^6 \\ &\quad - 86778s^5 + 57510s^4 + 43740s^3 - 19440s^2 - 10935s - 1215). \end{aligned}$$

The logarithmic derivative ansatz already gave expressions of the other coefficients in terms of  $a_3, b_1, b_2, d_1, d_2, e_1, q$ . With all coefficients parametrized, we find  $r_0 = 13824(5s + 1)(7s^3 - 3s^2 - s + 1)^5$ .

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